Quantum optics lecture
Part III: Coherent states, a laser driven atom and other bits and pieces

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Summary. In this lecture, we learn more about the quantised radiation field inside an optical cavity by considering coherent states. The following half of this lecture course will be devoted to the laser driven two-level atom placed inside the free radiation field. We start here with the introduction of some basic tools for analysing the time evolution of such a quantum system and first neglect spontaneous emission.

I. COHERENT STATES OF LIGHT

There are different bases one can choose for the description of the harmonic oscillator of the electromagnetic field inside a one-dimensional box with metallic sides. One basis, that is very convenient for most applications, is the *Fock basis* formed by the number states $|n\rangle$ with $n = 0, 1, 2, ...$ and

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,, \\ a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\,, \\ \langle n|m\rangle &= \delta_{n,m}\,. \end{aligned} \tag{1}$$

An alternative basis are the *coherent states*, whose properties most closely resemble those of classical states, like the output of a laser. They are defined as

$$|\alpha\rangle \equiv \exp\big(-\tfrac{1}{2}\,|\alpha|^2\big) \sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{2}$$

The number α can be any complex number. One can easily show that the states $|\alpha\rangle$ are normalised,

$$\langle\alpha|\alpha\rangle = \exp\big(-|\alpha|^2\big) \sum_{n=0}^\infty \frac{|\alpha|^{2n}}{n!}\,|n\rangle = 1\,. \tag{3}$$

Moreover, one can calculate the overlap between two different coherent states $|\alpha\rangle$ and $|\beta\rangle$,

$$\langle\alpha|\beta\rangle = \exp\big(-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)\big) \sum_{n=0}^\infty \frac{\alpha^{*n}\beta^n}{n!}\,|n\rangle = \exp\big(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2+\alpha^*\beta\big)\,, \tag{4}$$

which implies

$$\big|\,\langle\alpha|\beta\rangle\,\big|^2 \;=\; \exp\big(-|\alpha-\beta|^2\big)\,. \tag{5}$$

The coherent states are therefore not pairwise orthogonal. Since they can nevertheless be used to represent any state by writing it as a superposition of them, they are called an overcomplete basis. To find the decomposition for a given state $|\psi\rangle$ one should multiply this state with the unity operator $I\!I$ given by

$$I\!I = \sum_{n=0}^\infty |n\rangle\langle n| = \frac{1}{\pi}\int \mathrm{d}\alpha\,|\alpha\rangle\langle\alpha|\,. \tag{6}$$

To proof this relation one should calculate the intergral over the whole complex plane in the right hand side of Eq. (6) using for example the notation

$$\int \mathrm{d}\alpha\,|\alpha\rangle\langle\alpha| = \int_0^{2\pi}\mathrm{d}\varphi\int_0^\infty \mathrm{d}r\,|r\mathrm{e}^{\mathrm{i}\varphi}\rangle\langle r\mathrm{e}^{\mathrm{i}\varphi}| \tag{7}$$

and show that it equals π.

Coherent states are useful, since they are the eigenstates of the annihilation operator a with eigenvalue α,

$$a\,|\alpha\rangle = \exp\big(-\tfrac{1}{2}\,|\alpha|^2\big)\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle = \alpha\,|\alpha\rangle\,. \tag{8}$$

The mean number of photons for a system prepared in $|\alpha\rangle$ equals

$$\langle n\rangle_\alpha = \langle\alpha|\,a^\dagger a\,|\alpha\rangle = \|\,a\,|\alpha\rangle\,\|^2 = |\alpha|^2 \tag{9}$$

and the probability for finding m photons in the box is in this case given by

$$P_m = \big|\,\langle m|\alpha\rangle\,\big|^2 = \exp\big(-|\alpha|^2\big)\,\frac{|\alpha|^{2m}}{m!} = \exp\big(-\langle n\rangle_\alpha\big)\,\frac{\langle n\rangle_\alpha^m}{m!}\,. \tag{10}$$

As an example, calculate this probability distribution, i.e. P_m as a function of m, numerically for a small and a large value of m. For large m, you will see that the photon number distribution obeys a Poissonian statistics. The coherent states are the states whose electromagnetic fields resemble the possible classical solutions of the electromagnetic field inside an optical cavity most. For large α, there is only a small uncertainty between amplitude and phase of the **E**-field in the box.

II. TIME EVOLUTION AND SCHRÖDINGER EQUATIONS

Up to now, we were interested in observables and their expectation values for a given quantum state, like the mean photon number of a coherent state $|\alpha\rangle$. However, in general, the state of a quantum system is not constant in time and the system evolves according to the Schrödinger equation

$$\mathrm{i}\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = H\,|\psi(t)\rangle \quad\text{or}\quad \frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = -\frac{\mathrm{i}}{\hbar}\,H\,|\psi(t)\rangle\,, \tag{11}$$

where H is the corresponding Hamiltonian. We will use this equation, whose formal solution for time independent Hamiltonians H equals

$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle = \exp\Big(-\frac{\mathrm{i}}{\hbar}\,Ht\Big)\,|\psi(0)\rangle\,, \tag{12}$$

in the following to describe the time evolution of a laser driven two-level atom.

The operator $U(t,0)$ in Eq. (12) is called the *time evolution operator* and is a function of the Hamiltonian H. To see how to calculate the function of a Hermitian operator, let us consider again the operator $\mathbf{A}$ with eigenvalues λ_n and eigenstates $|n\rangle$ given by

$$\mathbf{A} = \sum_n \lambda_n\,|n\rangle\langle n|\,. \tag{13}$$

For this operator, one can easily calculate $\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A} = \sum_n \lambda_n^2\,|n\rangle\langle n|$ and $\mathbf{A}^m = \mathbf{A}\cdot\ldots\cdot\mathbf{A} = \sum_n \lambda_n^m\,|n\rangle\langle n|$, which implies the rule

$$f(\mathbf{A}) = \sum_n f(\lambda_n)\,|n\rangle\langle n|\,. \tag{14}$$

This relation allows to calculate the function of any Hermitian operator as long as its eigenvalues and eigenstates are known [1]. As an example, let us consider the Hamiltonian $H = \sum_n \hbar\omega_n\,|n\rangle\langle n|$ with the corresponding time evolution operator

$$U(t,0) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\,Ht\Big) = \sum_n \exp\big(-\mathrm{i}\omega_n t\big)\,|n\rangle\langle n|\,. \tag{15}$$

If the system is initially in the energy eigenstate $|\psi(0)\rangle = |m\rangle$, then $|\psi(t)\rangle = \exp\big(-\mathrm{i}\omega_m t\big)\,|m\rangle$. This shows, that the energy eigenstates of a system do not evolve in time and only accumulate a phase. Eq. (15) is actually one of the motivations, why people are interested in finding the energy eigenstates of a system.

Like in classical mechanics, it is not always possible to calculate the time evolution of a system directly and to solve the Schrödinger equation (11) without changing the reference frame first. In quantum mechanics, these reference frames are called pictures, with the usual one being the *Schrödinger picture*. In this picture the Schrödinger equation is the one given in Eq. (11), the time evolution operator is the one given in Eq. (12) and systems and observables are described by state vectors $|\psi\rangle$ and Hermitian operators $\mathbf{A}$, respectively.

An alternative is to describe quantum mechanical systems in the *interaction picture* with respect to a Hamiltonian H_0, which can be chosen arbitrarily. Suppose we define a Hamiltonian H_1 such that

$$H = H_0 + H_1 \tag{16}$$

and introduce the state vector $|\psi_{\rm I}(t)\rangle$ for the description of the system in the interaction picture such that

$$|\psi_{\rm I}(t)\rangle \equiv U_0^\dagger(t,0)\,|\psi(t)\rangle \tag{17}$$

with

$$U_0(t,0) = \exp\Big(-\frac{\rm i}{\hbar}\,H_0 t\Big) \;\text{ and }\; U_0^\dagger(t,0) = \exp\Big(\frac{\rm i}{\hbar}\,H_0 t\Big)\,. \tag{18}$$

Then one can show that

$$\frac{\rm d}{{\rm d}t}\,|\psi_{\rm I}(t)\rangle = \Big(\frac{\rm d}{{\rm d}t}U_0^\dagger(t,0)\Big)\,|\psi(t)\rangle + U_0^\dagger(t,0)\cdot\frac{\rm d}{{\rm d}t}\,|\psi(t)\rangle\,. \tag{19}$$

Using the above equations, this can be simplified to

$$\frac{\rm d}{{\rm d}t}\,|\psi_{\rm I}(t)\rangle = -\frac{\rm i}{\hbar}\,H_{\rm I}(t)\,|\psi_{\rm I}(t)\rangle \;\text{ with }\; H_{\rm I}(t) \equiv U_0^\dagger(t,0)\,H_1\,U_0(t,0)\,. \tag{20}$$

Here $H_{\rm I}$ is the so-called interaction Hamiltonian. The motivation for the introduction of the interaction picture is that, in many cases, one can find a Hamiltonian H_0 such that solving the Schrödinger equation (20) is much easier than solving the original equation (11). However, H_0 always has to be chosen such that one can easily calculate $U_0(t,0)$ and $U_0^\dagger(t,0)$.

Of course, there is no point in solving a simpler equation, if we do not know how to interpret the result. The expectation values of observables can be calculated in exactly the same way as in the usual Schrödinger picture, if we describe an observable $\mathbf{A}$ in the interaction picture by the transformed observable

$$\mathbf{A}_{\rm I}(t) \equiv U_0^\dagger(t,0)\,\mathbf{A}\,U_0(t,0)\,. \tag{21}$$

Then we have that

$$\langle\mathbf{A}\rangle_\psi = \langle\psi(t)|\,\mathbf{A}\,|\psi(t)\rangle = \langle\psi_{\rm I}(t)|\,\mathbf{A}_{\rm I}(t)\,|\psi_{\rm I}(t)\rangle\,. \tag{22}$$

Finally, we remark that $|\psi_{\rm I}(0)\rangle = |\psi(0)\rangle$ by definition (see Eq. (15)). This means, the initial state for the calculation of $|\psi_{\rm I}(t)\rangle$ is the same as in the Schrödinger picture.

Another reference frame that is often used in quantum optics is the *Heisenberg picture*. In the Schrödinger picture, only the states of a system evolve in time while the observables remain constant. In the interaction picture, state vectors and observables have to be calculated in order to make predictions for certain measurements at a given time t. However, in the Heisenberg picture, quantum mechanical systems are described by evolving their observables, while the state vector does not change in time. The Heisenberg picture is a special example of the interaction picture with

$$H \equiv H_0\,,\quad H_{\rm I}(t) \equiv 0\,,\quad \frac{\rm d}{{\rm d}t}\,|\psi_{\rm I}(t)\rangle = 0 \;\text{ and }\; A_{\rm I}(t) \equiv U(t,0)^\dagger\,A\,U(t,0)\,. \tag{23}$$

Obviously, changing from the Schrödinger picture into the Heisenberg picture does not simplify any calculations. In some cases, doing this is nevertheless useful, since predictions about a system can be made without specifying its initial state.

III. A LASER DRIVEN TWO-LEVEL ATOM

The laser driven two-level atom trapped inside the free radiation field, which became experimentally accessible since the development of the Paul trap by Wolfgang Paul in 1956 [2], will be the main subject of the remaining quantum

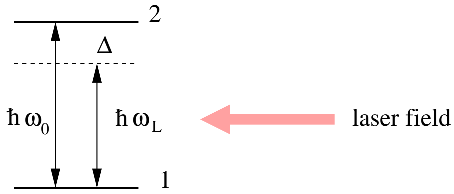

FIG. 1: Level scheme of a single two-level atom with energy level separation $\hbar\omega_0$ driven by a laser field with frequeny $\omega_{\rm L}$ and detuning $\Delta = \omega_0 - \omega_{\rm L}$.

optics course. Let us start by neglecting spontaneous emission. In the following, we consider a two-level atom with ground state $|1\rangle$ and excited state $|2\rangle$ as shown in Figure 1. The energy difference between the two states equals $\hbar\omega_0$. Aim is to calculate the effect of a laser field with frequency $\omega_{\rm L}$ onto the state of the atom given the initial state $|\psi(0)\rangle = |1\rangle$. To achieve this, we have to solve the Schrödinger equation (11) but first we have to find the corresponding Hamiltonian H. In the following, we treat the laser field in a classical way and assume that it generates an electromagnetic field with

$$\mathbf{E}_{\rm L}(t) = \mathbf{E}_0 \, {\rm e}^{-{\rm i}\omega_{\rm L} t} + \mathbf{E}_0^* \, {\rm e}^{{\rm i}\omega_{\rm L} t} \, , \tag{24}$$

at the position of the atom. Here $\mathbf{E}_0$ is the amplitude of the applied laser field. In principle, from a quantum mechanical point of view, the laser field should be described by a coherent state $|\alpha\rangle$. However, for large values of α, this would not yield any other conclusions than what we will find using Eq. (24). It would only complicate the calculations below.

One term in the Hamiltonian H in the Schrödinger picture is the Hamiltonian describing the energy of the atomic states given by

$$H_{\rm atom} = \tfrac{1}{2}\hbar\omega_0 \left(|2\rangle\langle 2| - |1\rangle\langle 1| \right) . \tag{25}$$

Here we assumed, without loss of generality, that the state $|1\rangle$ has the energy $-\frac{1}{2}\hbar\omega_0$, while the excited state $|2\rangle$ has the energy $\frac{1}{2}\hbar\omega_0$. The term describing the atom-laser interaction equals in analogy to the classical expression for the energy of a single electron moving in a classical electromagnetic field

$$H_{\rm laser} = e \, \mathbf{E}_{\rm L}(t) \cdot \mathbf{x} \, , \tag{26}$$

where $\mathbf{x}$ is the quantum mechanical position operator. As one can see from this equation, the interaction between the atom and the laser is a dipole interaction. To bring the Hamiltonian (26) in a form, which is useful for us, we do the following trick and write

$$\begin{aligned} \mathbf{x} = I\!I \cdot \mathbf{x} \cdot I\!I &= \left(|1\rangle\langle 1| + |2\rangle\langle 2| \right) \mathbf{x} \left(|1\rangle\langle 1| + |2\rangle\langle 2| \right) \\ &= \langle 1|\,\mathbf{x}\,|2\rangle \, |1\rangle\langle 2| + \langle 2|\,\mathbf{x}\,|1\rangle \, |2\rangle\langle 1| \\ &= \mathbf{D} \, |1\rangle\langle 2| + \mathbf{D}^* \, |2\rangle\langle 1| \, . \end{aligned} \tag{27}$$

Here we used the fact that the energy eigenstates $|1\rangle$ and $|2\rangle$ of an atom are (due to the symmetry of their potential) always either symmetric or antisymmetric such that the matrix elements $\langle 1|\,\mathbf{x}\,|1\rangle = \langle 2|\,\mathbf{x}\,|2\rangle = 0$ (see also remark [3] at the end of the text). Moreover, we introduced the dipole moment $\mathbf{D} = \langle 1|\,\mathbf{x}\,|2\rangle$ characterising the dipole moment of the 1-2 transition of the atom. The calculation in Eq. (27) is known as the second quantisation of the atomic states and yields the Hamiltonian

$$\begin{aligned} H_{\rm laser} &= e \left(\mathbf{E}_0 \, {\rm e}^{-{\rm i}\omega_{\rm L} t} + \mathbf{E}_0^* \, {\rm e}^{{\rm i}\omega_{\rm L} t} \right) \cdot \left(\mathbf{D} \, |1\rangle\langle 2| + \mathbf{D}^* \, |2\rangle\langle 1| \right) \\ &= e \, \mathbf{E}_0 \cdot \mathbf{D} \, {\rm e}^{-{\rm i}\omega_{\rm L} t} \, |1\rangle\langle 2| + e \, \mathbf{E}_0 \cdot \mathbf{D}^* \, {\rm e}^{-{\rm i}\omega_{\rm L} t} \, |2\rangle\langle 1| + e \, \mathbf{E}_0^* \cdot \mathbf{D} \, {\rm e}^{{\rm i}\omega_{\rm L} t} \, |1\rangle\langle 2| + e \, \mathbf{E}_0^* \cdot \mathbf{D}^* \, {\rm e}^{{\rm i}\omega_{\rm L} t} \, |2\rangle\langle 1| \, . \end{aligned} \tag{28}$$

The total Hamiltonian of the laser driven atom equals in the Schrödinger picture

$$H = H_{\rm atom} + H_{\rm laser} \, . \tag{29}$$

Due to the time dependence of Eq. (28), solving the corresponding Schrödinger equation is not straightforward but we can now simplify our life by using the interaction picture introduced in the previous section.

For reasons that become obvious later, let us define H_0 as

$$H_0 = \tfrac{1}{2}\hbar\omega_{\rm L} \left(|2\rangle\langle 2| - |1\rangle\langle 1| \right) \tag{30}$$

with the corresponding time evolution operators

$$\begin{aligned} U_0(t,0) &= \exp\left(\tfrac{\rm i}{2}\omega_{\rm L} t \right) |1\rangle\langle 1| + \exp\left(-\tfrac{\rm i}{2}\omega_{\rm L} t \right) |2\rangle\langle 2| \, , \\ U_0^\dagger(t,0) &= \exp\left(-\tfrac{\rm i}{2}\omega_{\rm L} t \right) |1\rangle\langle 1| + \exp\left(\tfrac{\rm i}{2}\omega_{\rm L} t \right) |2\rangle\langle 2| \, . \end{aligned} \tag{31}$$

Using Eq. (20), we find that the Hamiltonian of the laser driven atom becomes in the interaction picture

$$\begin{aligned} H_{\rm I}(t) &\equiv U_0^\dagger(t,0) \left(H - H_0 \right) U_0(t,0) \\ &= \tfrac{1}{2}\hbar(\omega_0 - \omega_{\rm L}) \left(|2\rangle\langle 2| - |1\rangle\langle 1| \right) + e\,\mathbf{E}_0 \cdot \mathbf{D}\, {\rm e}^{-2{\rm i}\omega_{\rm L} t} \, |1\rangle\langle 2| + e\,\mathbf{E}_0 \cdot \mathbf{D}^* \, |2\rangle\langle 1| \\ &\quad + e\,\mathbf{E}_0^* \cdot \mathbf{D} \, |1\rangle\langle 2| + e\,\mathbf{E}_0^* \cdot \mathbf{D}^* \, {\rm e}^{2{\rm i}\omega_{\rm L} t} \, |2\rangle\langle 1| \, . \end{aligned} \tag{32}$$

Experience shows that the contribution of fast oscillating terms in a Hamiltonian to the time evolution of a system is in general negligible compared to terms that are constant in time. This approximation is known as the *rotating wave approximation.* Applying it here, we find

$$H_{\rm I}(t) = H_{\rm I} = \tfrac{1}{2}\hbar\Delta \left(|2\rangle\langle 2| - |1\rangle\langle 1| \right) + \tfrac{1}{2}\hbar\Omega \, |2\rangle\langle 1| + \tfrac{1}{2}\hbar\Omega^* \, |1\rangle\langle 2| \, , \tag{33}$$

with the Rabi frequency Ω and the detuning Δ defined as

$$\Omega \equiv \frac{2e\,\mathbf{E}_0 \cdot \mathbf{D}^*}{\hbar} \quad \text{and} \quad \Delta \equiv \omega_0 - \omega_{\rm L} \, . \tag{34}$$

Note that the Rabi frequency Ω is a measure for the amplitude of the applied laser field although it is called a frequency. The motivation for the choice of the interaction picture as above was to obtain a time independent Hamiltonian. We now have all the tools to calculate the time evolution of the atom in the presence of a detuned laser field. We will do this at the beginning of the next lecture. Just one final remark. The Rabi frequency Ω can be considered as real, since we can include any possible phase factor ${\rm e}^{{\rm i}\varphi}$ into the definition of the state $|2\rangle$. This means, we replace $|2\rangle$ by ${\rm e}^{{\rm i}\varphi}\,|2\rangle$, which has no other consequences than simplifying our calculations, since global phase factors have no physical effect.

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[1] We see later that it is enough to know the eigenvalues of $\mathbf{A}$.

[2] Dehmelt and Paul won the Nobel price for their contribution to the development of ion trap techniques and laser cooling in 1989 together with Ramsey. Trapping of single ions (and atoms) has many applications, like precision spectroscopy and quantum computing, but is especially exciting since it allows to directly observe quantum mechanical effects on the single atom level instead of observing quantum mechanics indirectly as this is the case in experiments observing for example the black body radiation.

[3] This is the case, since the application of the position operator $\mathbf{x}$ changes a symmetric eigenstate into an antisymmetric one and vice versa, which has no overlap with the original state.