Quantum optics lecture
Part II: The second quantisation of the free radiation field

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Summary. This is probably the conceptually most difficult lecture of the whole course. We will discuss the quantum description of the free radiation field. We ask again the question, what is a photon and make a short excursion into quantum field theory. To learn more about the second quantisation, I recommend the book *The Quantum Theory of Light* by R. Loudon.

I. AN INTUITIVE PICTURE

To get some intuition of what is meant by a photon, we first consider the solutions of the Maxwell equations for the electromagnetic field $\mathbf{E}(\mathbf{r},t)$ inside a superconducting box. Such a box with equal length L of all its sides (see Figure 1) is called an optical cavity. Its metallic walls introduce boundary conditions, which demand that the electric field $\mathbf{E}(\mathbf{r},t)$ vanishes along their surface. This is a result of the fact that any $\mathbf{E}$-field of this type is always immediately compensated by a current within the surface. In addition, to the boundary conditions, the electric field has to fulfil the equations

$$\begin{aligned} \nabla^2 \, \mathbf{E}(\mathbf{r},t) &= \frac{1}{c^2} \frac{\delta^2}{\delta t^2} \, \mathbf{E}(\mathbf{r},t) \, , \\ \nabla \cdot \mathbf{E}(\mathbf{r},t) &= 0 \, . \end{aligned} \tag{1}$$

which are a consequence of Maxwell's equations. A complete set of solutions to this is given by the $\mathbf{E}$-fields with the x, y and z-components

$$\begin{aligned} E_x(\mathbf{r},t) &= E_x(t) \, \cos(k_x x) \, \sin(k_y y) \, \sin(k_z z) \, , \\ E_y(\mathbf{r},t) &= E_y(t) \, \sin(k_x x) \, \cos(k_y y) \, \sin(k_z z) \, , \\ E_z(\mathbf{r},t) &= E_z(t) \, \sin(k_x x) \, \sin(k_y y) \, \cos(k_z z) \, . \end{aligned} \tag{2}$$

They fulfil the boundary conditions, like $E_x(\mathbf{r},t) = 0$ for $y = 0$ and $y = L$, if

$$k_x = \frac{\pi n_x}{L} \, , \; k_y = \frac{\pi n_y}{L} \; \text{ and } \; k_z = \frac{\pi n_z}{L} \; \text{ with } \; n_x, \, n_y, \, n_z = 0, \, 1, \, 2, \, ... \tag{3}$$

and Eqs. (1), if the amplitude $\mathbf{E}(t) = (E_x(t), E_y(t), E_z(t))^{\mathrm{T}}$ obeys the differential equation

$$\frac{\delta^2}{\delta t^2} \, \mathbf{E}(t) = -\omega_k^2 \, \mathbf{E}(t) \; \text{ with } \; \omega_k = ck \, . \tag{4}$$

The possible solutions of the electromagnetic field inside the optical cavity are superpositions of all the different solutions for the electromagnetic waves in the x, y and z-direction given above. Each direction can be described by an infinite set of possible solutions in form of standing waves. They are given in Eq. (2) and are each characterised by a discrete wave number k_x, k_y or k_z and by the amplitude E_x, E_y or E_z, respectively.

We can use these considerations now to get an intuitive picture of the *free radiation field*, which denotes the vacuum in which an atom might be placed and emits photons into. One possibility is to imagine the free radiation field as a

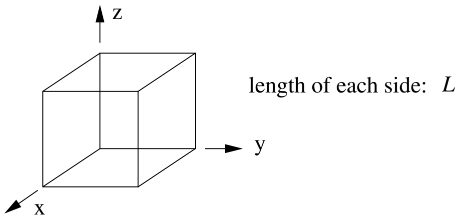

FIG. 1: The optical cavity, that we consider here, is a box with metallic surfaces with width, height and depth L.

large optical cavity ($L \to \infty$) which can be populated by standing wave solutions of the electromagnetic field. This electromagnetic field is made up of photons, whose energy is proportional k_x, k_y or k_z, respectively [1], and whose maximum amplitude $E_x(t)$, $E_y(t)$ or $E_z(t)$ relates to the number of photons contained in the box in the x, y or z-direction. The real quantum description of the electromagnetic field arises, when we treat the field inside the optical cavity as quantum mechanical harmonic oscillators, whose possible eigenstates have the energies $\hbar\omega_k(n_{\mathbf{k},\lambda} + \frac{1}{2})$. We see later that $n_{\mathbf{k},\lambda}$ is the number of photons with wave vector $\mathbf{k}$ and polarisation λ.

II. THE QUANTUM MECHANICAL HARMONIC OSCILLATOR

Before we discuss, how the electromagnetic field inside a box can be treated as a harmonic oscillator, let us first have a look at a particle with momentum operator $\mathbf{p}$ and position operator $\mathbf{x}$ in a harmonic potential. Its Hamiltonian is given by

$$H = \frac{\mathbf{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \, \mathbf{x}^2 \quad \text{with} \quad [\mathbf{x}, \mathbf{p}] = \mathrm{i}\hbar \, . \tag{5}$$

One convinient way of finding the eigenstates of this Hamiltonian is to introduce the operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega \mathbf{x} + \mathrm{i} \, \mathbf{p} \right) , \\ a^\dagger &= \frac{1}{\sqrt{2m\hbar\omega}} \left(m\omega \mathbf{x} - \mathrm{i} \, \mathbf{p} \right) . \end{aligned} \tag{6}$$

Using this notation, the position and momentum operators become

$$\begin{aligned} \mathbf{x} &= \sqrt{\frac{\hbar}{2m\omega}} \, (a + a^\dagger) \, , \\ \mathbf{p} &= \mathrm{i} \sqrt{\frac{m\hbar\omega}{2}} \, (a - a^\dagger) \, . \end{aligned} \tag{7}$$

The reason for introducing a and $a^\dagger$ is that the Hamiltonian (5) can now be written in a simpler form. This can be shown by calculating $a^\dagger a$ and $aa^\dagger$ using the commutator relation given in Eq. (1), which yields

$$\begin{aligned} a^\dagger a &= \frac{1}{\hbar\omega} \, H - \tfrac{1}{2} \, , \\ aa^\dagger &= \frac{1}{\hbar\omega} \, H + \tfrac{1}{2} \, . \end{aligned} \tag{8}$$

From this we conclude that

$$H = \hbar\omega (a^\dagger a + \tfrac{1}{2}) \quad \text{with} \quad [a, a^\dagger] = 1 \, . \tag{9}$$

This representation of the Hamiltonian and the commutator relation for a and $a^\dagger$ can now be used to find the energy eigenvalues and eigenvectors of the harmonic oscillator of a single particle.

Suppose, $|n\rangle$ is an eigenstate of H with energy E_n. Calculating $H \, a^\dagger |n\rangle$, one can then show that $a^\dagger |n\rangle$ is also an eigenstate of H with eigenvalue $E_n + \hbar\omega$. Using Eq. (9), we find indeed

$$\begin{aligned} H \, a^\dagger |n\rangle &= \hbar\omega (a^\dagger a + \tfrac{1}{2}) \, a^\dagger |n\rangle \\ &= \hbar\omega (a^\dagger a a^\dagger + \tfrac{1}{2} a^\dagger) \, |n\rangle \\ &= \hbar\omega (a^\dagger a^\dagger a + a^\dagger + \tfrac{1}{2} a^\dagger) \, |n\rangle \\ &= a^\dagger \, H|n\rangle + \hbar\omega \, a^\dagger |n\rangle \\ &= (E_n + \hbar\omega) \, a^\dagger |n\rangle \, . \end{aligned} \tag{10}$$

Analogously, namely by calculating $H \, a|n\rangle$, one can show that $a|n\rangle$ is an eigenstate of H with eigenvalue $E_n - \hbar\omega$. As a consequence, the energy eigenstates of the harmonic oscillator form a ladder system as shown in Figure 2 with energy spacing $\hbar\omega$.

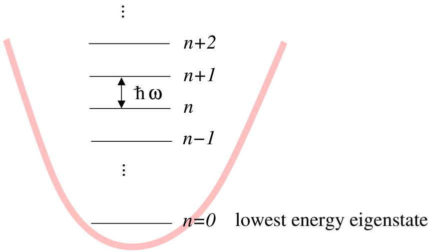

FIG. 2: Level diagram of a single particle inside a harmonic oscillator potential. Later, we compare this harmonic oscillator with the optical cavity introduced in Section I, identify the ground state with the photon vacuum state and interpret the state $|n\rangle$ as an n-photon state.

In the following, we denote the state with the lowest energy as $|0\rangle$. Such a state must exist and must fulfil the condition

$$a\,|0\rangle = 0\,. \tag{11}$$

Moreover, we can deduce

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,, \\ a^\dagger\,|n-1\rangle &= \sqrt{n}\,|n\rangle \end{aligned} \tag{12}$$

from the fact that $a^\dagger a\,|n\rangle = n\,|n\rangle$ and $\langle n|\,a^\dagger a\,|n\rangle = n$. The operator $n = a^\dagger a$ is therefore also called the *number operator*. Since the operator $a^\dagger$ applied to $|n\rangle$ creates a state with one more energy quanta $\hbar\omega$, it is called the *creation operator*, while a, doing the inverse, is the *annihilation operator*.

III. QUANTISATION OF THE FREE RADIATION FIELD

The free radiation field is the quantised electromagnetic field inside an optical cavity with $L \to \infty$ in the absence of any charges and currents. Experimental observations suggest that we should identify its vacuum state (no photons) with the ground state $|0\rangle$ of a collection of harmonic oscillators while its number states correspond to states with an integer number of photons in the metallic box. As we see in the following, one way of achieving this is to identify the amplitudes of the vector potential of the electromagnetic field $\mathbf{A}(\mathbf{r},t)$ with the annihilation operators of harmonic oscillators.

Let us now have a look at Maxwell's equations in the absence of currents and charges for the magnetic field $\mathbf{B}(\mathbf{r},t)$ and the transversal electric field $\mathbf{E}_\mathrm{T}(\mathbf{r},t)$, given by

$$\begin{aligned} \nabla \times \mathbf{E}_\mathrm{T} &= -\frac{\delta}{\delta t}\,\mathbf{B}\,, \\ \nabla \cdot \mathbf{E}_\mathrm{T} &= 0\,, \\ \frac{1}{\mu_0}\,\nabla \times \mathbf{B} &= \epsilon_0\,\frac{\delta}{\delta t}\,\mathbf{E}_\mathrm{T}\,, \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{13}$$

with

$$c = 1/\sqrt{\epsilon_0 \mu_0}\,. \tag{14}$$

One way of solving these equations is to introduce the vector potential $\mathbf{A}(\mathbf{r},t)$ with

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E}_\mathrm{T} = -\frac{\delta}{\delta t}\,\mathbf{A}\,. \tag{15}$$

Then most of the Maxwell equations (13) are automatically fulfilled. Conveniently, there is some freedom in choosing $\mathbf{A}(\mathbf{r},t)$ without affecting the solutions $\mathbf{B}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$. Here we use the Coulomb gauge and demand $\nabla \cdot \mathbf{A} = 0$, which yields the wave equation

$$\frac{1}{c^2}\,\frac{\delta^2}{\delta t^2}\,\mathbf{A} - \nabla^2\,\mathbf{A} = 0\,. \tag{16}$$

Proceeding as in Section I, one can show that the general solution to this equation is the vector potential

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}} \sum_{\lambda=1,2} \underline{\epsilon}_{\mathbf{k},\lambda}\, A_{\mathbf{k},\lambda}(\mathbf{r},t) \tag{17}$$

with the unit vectors $\underline{\epsilon}_{\mathbf{k},1}$ and $\underline{\epsilon}_{\mathbf{k},2}$ defined with respect to $\mathbf{k}$ such that

$$\begin{aligned} \underline{\epsilon}_{\mathbf{k},\lambda} \cdot \mathbf{k} &= 0 \;\text{ for }\; \lambda = 1,\, 2\,, \\ \underline{\epsilon}_{\mathbf{k},1} \cdot \underline{\epsilon}_{\mathbf{k},2} &= 0\,. \end{aligned} \tag{18}$$

Moreover, one should have

$$A_{\mathbf{k},\lambda}(\mathbf{r},t) = A_{\mathbf{k},\lambda}\, \mathrm{e}^{-\mathrm{i}(\omega_k t - \mathbf{k}\cdot\mathbf{r})} + A^*_{\mathbf{k},\lambda}\, \mathrm{e}^{\mathrm{i}(\omega_k t - \mathbf{k}\cdot\mathbf{r})}\,, \tag{19}$$

where the possible components of $\mathbf{k}$ are all of the form

$$\mathbf{k} = \Big(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L}\Big) \;\text{ with }\; n_x,\, n_y,\, n_z = 0,\, \pm 1,\, \pm 2,\, ... \tag{20}$$

Up to here, we haven't treated the system in any non-classical way. Eq. (17) simply is the vector potential of electromagnetic waves with wave vectors $\mathbf{k}$, polarisation $\lambda = 1$ or 2 and polarisation vectors $\underline{\epsilon}_{\mathbf{k},1}$ and $\underline{\epsilon}_{\mathbf{k},2}$. The strength of the potential and therefore also the amplitude of the corresponding electromagnetic fields is given by the amplitudes $A_{\mathbf{k},\lambda}$ on the right hand side of Eq. (19).

The second quantisation of the free radiation field is to identify these oscillating variables with the annihilation operators of harmonic oscillators. Each possible mode $(\mathbf{k},\lambda)$ of the free radiation field constitutes one harmonic oscillator and we denote its respective annihilation operator as $a_{\mathbf{k},\lambda}$ and replace

$$\begin{aligned} A_{\mathbf{k},\lambda}\, \mathrm{e}^{-\mathrm{i}\omega_k t} &\longrightarrow \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}}\, a_{\mathbf{k},\lambda}\,, \\ A^*_{\mathbf{k},\lambda}\, \mathrm{e}^{\mathrm{i}\omega_k t} &\longrightarrow \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}}\, a^\dagger_{\mathbf{k},\lambda}\,. \end{aligned} \tag{21}$$

We see below, why the factor in front of the annihilation and creation operators $a_{\mathbf{k},\lambda}$ and $a^\dagger_{\mathbf{k},\lambda}$ have been chosen in exactly this form. Note that the operators $a_{\mathbf{k},\lambda}$ and $a^\dagger_{\mathbf{k},\lambda}$ fulfil the commutator relations

$$[\, a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\,] = \delta_{\mathbf{k}-\mathbf{k}'}\, \delta_{\lambda,\lambda'} \tag{22}$$

and there is no coupling between different field modes $(\mathbf{k},\lambda)$. We see later, that this seemingly random choice of the photon annihilation and creation operators yields field operators and Hamiltonians that are in good agreement with experimental observations. They form the basis for any quantum optical description of light-matter interactions.

Finally, let us summarise the quantum mechanical operators for the vector potential, the electric and magnetic field and the Hamiltonian of the free radiation field $H_{\rm env}$. In direct analogy to Eqs. (17) and (19) we find that the observable for the vector potential equals

$$A_{\mathbf{k},\lambda}(\mathbf{r}) \;=\; \sum_{\mathbf{k}} \sum_{\lambda=1,2} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}} \left[\underline{\epsilon}_{\mathbf{k},\lambda}\, a_{\mathbf{k},\lambda}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \underline{\epsilon}^*_{\mathbf{k},\lambda}\, a^*_{\mathbf{k},\lambda}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right]. \tag{23}$$

Using Eq. (15), we can then calculate the quantum field theoretical observable for the electric and magnetic field and obtain

$$\begin{aligned} \mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r}) &= \mathrm{i} \sum_{\mathbf{k}} \sum_{\lambda=1,2} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}} \left[\underline{\epsilon}_{\mathbf{k},\lambda}\, a_{\mathbf{k},\lambda}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \underline{\epsilon}_{\mathbf{k},\lambda}\, a^*_{\mathbf{k},\lambda}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right], \\ \mathbf{B}_{\mathbf{k},\lambda}(\mathbf{r}) &= \mathrm{i} \sum_{\mathbf{k}} \sum_{\lambda=1,2} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k L^3}} \left[\mathbf{k} \times \underline{\epsilon}_{\mathbf{k},\lambda}\, a_{\mathbf{k},\lambda}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \mathbf{k} \times \underline{\epsilon}_{\mathbf{k},\lambda}\, a^*_{\mathbf{k},\lambda}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right]. \end{aligned} \tag{24}$$

It is important to note that these operators are time independent operators. The reason is that the observables for the quantised electromagnetic field should be at any time the same. In general, only the state vector of a system changes in time.

Finally, let us calculate the Hamiltonian describing the energy of the light inside an optical cavity of length L. Using the classical relation

$$E_{\rm env} \;=\; \tfrac{1}{2} \int_V {\rm d}\mathbf{x} \left[\epsilon_0 \, |\mathbf{E}_{\mathbf{k},\lambda}(\mathbf{r})|^2 + \frac{1}{\mu_0} \, |\mathbf{B}_{\mathbf{k},\lambda}(\mathbf{r})|^2 \, \right] \tag{25}$$

and the correspondence principle (i.e. simply replace the fields in the above equation by the corresponding quantum mechanical operators) one can, in principle (we don't do this calculation here, since this would be too time consuming), derive the Hamiltonian for the free radiation field. The factors in Eqs. (21) are chosen such that we indeed obtain an oscillator for a collection of harmonic oscillators and find

$$H_{\rm env} \;=\; \sum_{\mathbf{k}} \sum_{\lambda=1,2} \hbar\omega_k \left[a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} + \tfrac{1}{2} \right] . \tag{26}$$

For most purposes, one can ignore the offset (last term in this equation) and simply consider the Hamiltonian

$$H_{\rm env} \;=\; \sum_{\mathbf{k}} \sum_{\lambda=1,2} \hbar\omega_k \, a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} \tag{27}$$

as the Hamiltonian of the free radiation field. Note that we are interested in the regime, where L is large compared to all other relevant wave length in the setup, so that the possible photon frequencies ω_k become continuous. All kinds of photons become allowed. The most important results of this lecture are the Eqs. (22), (24) and (27).

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[1] Remember that the energy of a single photon is indeed proportional to its wave vector, i.e. $E = \hbar\omega_k$ with $\omega_k = ck$.