

# Quantum optics lecture

## Part IV: Spontaneous emission and Einstein's rate equations

Almut Beige

(Dated: March 23, 2005)

**Summary.** Today, we explicitly calculate the time evolution of a laser driven two-level atom. Afterwards we discuss the easiest model for the description of this system in the presence of spontaneous emission and introduce *Einstein's rate equations*. We also prepare for the final lecture, where we introduce the quantum optical description of the same system from basic quantum mechanical principles with the quantum jump approach and with master equations.

### I. CONTINUATION OF PART III

In the last lecture we derived the Hamiltonian for a laser-driven two-level atom in the interaction picture and found that it equals

$$H_I = \frac{1}{2}\hbar\Omega (|2\rangle\langle 1| + |1\rangle\langle 2|) + \frac{1}{2}\hbar\Delta (|2\rangle\langle 2| - |1\rangle\langle 1|), \quad (1)$$

where  $\Omega$  and  $\Delta$  are the Rabi frequency and the detuning of the applied laser field respectively. The states  $|1\rangle$  and  $|2\rangle$  denote the ground and excited state of the atom. If we change into a matrix and vector representation, we can write

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad H_I = \frac{1}{2}\hbar \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} \equiv \frac{1}{2}\hbar \mathbf{M}. \quad (2)$$

The next step is to calculate the corresponding time evolution operator  $U_I(t, 0)$ . From the previous lecture (see Eqs. (14) and (15) there) we know that

$$U_I(t, 0) = \exp\left(-\frac{i}{\hbar}H_I t\right) = \exp\left(-\frac{i}{2}\mathbf{M}t\right) = \sum_n \exp\left(-\frac{i}{2}\lambda_n t\right) |\lambda_n\rangle\langle\lambda_n|, \quad (3)$$

if the  $\lambda_n$  and  $|\lambda_n\rangle$  are the eigenvalues and eigenstates of  $\mathbf{M}$ . Here we use an alternative but identical expression for the time evolution operator, namely the equation

$$U_I(t, 0) = \frac{\mathbf{M} - \lambda_2}{\lambda_1 - \lambda_2} \exp\left(-\frac{i}{2}\lambda_1 t\right) + \frac{\mathbf{M} - \lambda_1}{\lambda_2 - \lambda_1} \exp\left(-\frac{i}{2}\lambda_2 t\right), \quad (4)$$

which allows us to avoid having to find the eigenvectors of  $\mathbf{M}$  but gives the same result when applied to an arbitrary initial atomic state  $|\psi(0)\rangle = c_1 |\lambda_1\rangle + c_2 |\lambda_2\rangle$ . The eigenvalues of  $\mathbf{M}$  are

$$\lambda_{1,2} = \pm\sqrt{\Delta^2 + \Omega^2} \equiv \pm R, \quad (5)$$

which yields

$$\begin{aligned} U_I(t, 0) &= \frac{\mathbf{M}}{2R} \left[ \exp\left(-\frac{i}{2}Rt\right) - \exp\left(\frac{i}{2}Rt\right) \right] + \frac{1}{2} \left[ \exp\left(-\frac{i}{2}Rt\right) + \exp\left(\frac{i}{2}Rt\right) \right] \\ &= \cos\left(\frac{1}{2}Rt\right) - \frac{i\mathbf{M}}{R} \sin\left(\frac{1}{2}Rt\right) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\left(\frac{1}{2}Rt\right) - \frac{i}{R} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} \sin\left(\frac{1}{2}Rt\right). \end{aligned} \quad (6)$$

Suppose, the atom is initially prepared in its ground state and  $|\psi(0)\rangle = |1\rangle$ . Then the state of the system equals at  $t$

$$|\psi_I(t)\rangle = U_I(t, 0) |1\rangle = \left[ \cos\left(\frac{1}{2}Rt\right) + \frac{i\Delta}{R} \sin\left(\frac{1}{2}Rt\right) \right] |1\rangle - \frac{i\Omega}{R} \sin\left(\frac{1}{2}Rt\right) |2\rangle. \quad (7)$$

The amount of population in level 2 at  $t$  is therefore given by

$$P_2(t) = |\langle 2|\psi_I(t)\rangle|^2 = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2\left(\frac{1}{2}Rt\right). \quad (8)$$

In the presence of a laser field, the state of the atom starts to do so-called *Rabi oscillations*. As can be seen from Eq. (8), the laser field creates a potential in which the electron starts oscillating between the ground state and the excited state. In case of a resonant laser field ( $\Delta = 0$ ), the initial population in the ground state can be transferred completely into the excited state and  $P_2$  can become 1 (see Figure 1). For a detuned laser field, the maximum population in  $|2\rangle$  remains always smaller than 1.

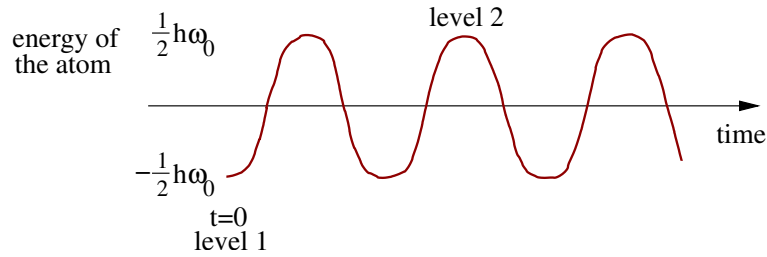


FIG. 1: Sketch of the time evolution of a two-level atom driven by a resonant laser field ( $\Delta = 0$ ).

## II. EINSTEIN'S RATE EQUATIONS

The simplest model for the description of a laser driven atom inside the free radiation field are Einstein's rate equations. They are already a very good model to predict the behaviour of such a system as it is observed in quantum optical experiments with trapped single atoms or ions. The basic assumption in this model is that the atom is always either in level 1 or in level 2. The role of the laser is to induce sudden jumps from one level to the other. We have seen in the previous Section that this is at least not too far away from the truth (see Figure 1). The probability density for such a jump is proportional to the population in the respective level. Moreover, a spontaneous emission of a photon can occur at any time with a probability density proportional to the population in the excited state, which always results in a transition from level 2 to the ground state  $|1\rangle$ .

Einstein noticed that one can characterise the transitions between the two states by constant rates. The rate  $B$  describes in the following the effect of the laser field on the atom, which is often called an induced absorption or induced emission of a single photon from and into the laser field respectively, although, as we have seen above, this is not a very good interpretation of the effect of the laser. Einstein also introduced the  $A$ -coefficient as the decay rate of the excited state. It denotes the probability density for the spontaneous emission of a photon by an atom in the excited state. Using the  $A$  and  $B$  rates, one could use a random number generator to calculate a possible single trajectory for the time evolution of the atom as it is shown in Figure 2.

Suppose, we observe a whole ensemble of single atoms, each driven by a laser field and each randomly emitting photons, and not a single trajectory of an atom. Picking one atom out of the ensemble randomly, we can ask, what is the probability to find this atom in a certain state  $|i\rangle$ . This probability equals the population of level  $i$  averaged over the whole ensemble and we denote it in the following as usual by  $P_i$ . The populations  $P_i$  change according to the differential equations

$$\begin{aligned}\frac{d}{dt} P_1 &= -B P_1 + B P_2 + A P_2, \\ \frac{d}{dt} P_2 &= -B P_2 + B P_1 - A P_2,\end{aligned}\tag{9}$$

which are called *Einstein's rate equation*.

One feature already included in this simple model based on constant rates is the fact that immediately after an emission, the probability for a second photon emission is 0. The reason for this *antibunching* of the photon emission times is the fact that the atom is after an emission in  $|1\rangle$  and has to be transferred back into  $|2\rangle$  before the next emission can take place. Testing, whether this is the case is a useful experimental test, whether certain light comes indeed only from a single atom and not two or more. For many atoms, the probability for a photon emission is always the same and the distribution of the emission times is completely random. The extreme case, in which a subsequently emitted photon occurs on average earlier than in a completely random distribution with the same mean time between two emissions is called *bunching* of photons.

The *stationary state* of the atom averaged over a large ensemble of possible trajectories is the state, where the knowledge about the initial state of the system is lost and the average populations  $P_1$  and  $P_2$  change no longer in time such that

$$\frac{d}{dt} P_1 = \frac{d}{dt} P_2 = 0.\tag{10}$$

Calculating  $P_1$  and  $P_2$  using this equation and the additional condition

$$P_1 + P_2 = 1\tag{11}$$

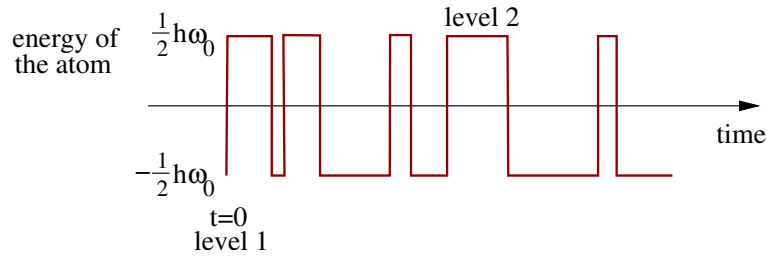


FIG. 2: A possible trajectory of a two-level atom as predicted by Einstein's rate equations. At time  $t = 0$ , the atom is in its ground state but can go over in the excited state due to the presence of a laser field. Once in the excited state, the atom returns in the ground state either via a spontaneous or the induced emission of a photon.

gives us the probability to find an atom after a long time in level 1 or 2 respectively. From Eqs. (10) and (11) we obtain the stationary state distribution

$$P_1 = \frac{A+B}{A+2B} \quad \text{and} \quad P_2 = \frac{B}{A+2B}. \quad (12)$$

In the limit of very weak laser driving, the atom would most likely be found in the ground state since  $P_1 = 1$  and  $P_2 = 0$  for  $B/A \rightarrow 0$ . For strong driving, at most half of the atomic population accumulates in the excited state, since  $P_1 = P_2 = \frac{1}{2}$  for  $B/A \rightarrow \infty$ .

### III. PERTURBATION THEORY FOR TIME DEPENDENT HAMILTONIANS

To find a more precise description of the atom, we will need another quantum mechanical tool for the calculation of time evolution operators. Suppose we have to solve the Schrödinger equation

$$\frac{d}{dt} |\psi_I(t)\rangle = -\frac{i}{\hbar} H_I(t) |\psi_I(t)\rangle. \quad (13)$$

Since  $|\psi_I(t)\rangle = U_I(t, 0) |\psi_I(0)\rangle$ , this equation is equivalent to

$$\frac{d}{dt} U_I(t, 0) = -\frac{i}{\hbar} H_I(t) U_I(t, 0), \quad (14)$$

which yields after integrating over both sides from 0 to  $t$

$$U_I(t, 0) - U_I(0, 0) = -\frac{i}{\hbar} \int_0^t dt' H_I(t') U_I(t', 0). \quad (15)$$

We know that  $U_I(0, 0) = \mathbb{I}$ , the unity operator. Moreover, using Eq. (15) several times to calculate the time evolution operator on its right hand side, we then find that

$$U_I(t, 0) = \mathbb{I} - \frac{i}{\hbar} \int_0^t dt' H_I(t') - \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} dt'' H_I(t') H_I(t'') + \mathcal{O}(t^3). \quad (16)$$

The derivation of this formulae is called *second order perturbation theory* since we evolved the time evolution operator up to the second order in  $t$ . For any time independent Hamiltonian, the final result immediately simplifies to

$$U_I(t, 0) = \exp\left(-\frac{i}{\hbar} H_I t\right), \quad (17)$$

as we know from before.

### IV. THE HAMILTONIAN OF AN ATOM IN THE FREE RADIATION FIELD

Before we can have a look at the physical processes that result in the spontaneous generation of a single photon, we have to derive the Hamiltonian for the interaction between the atom and the free radiation field. As before in the

derivation of the laser Hamiltonian, this interaction is a dipole interaction of the form

$$H_{\text{int}} = e \mathbf{E}(\mathbf{r} = 0) \cdot \mathbf{x}, \quad (18)$$

where  $\mathbf{x}$  is the quantum mechanical position operator

$$\mathbf{x} = \mathbf{D} |1\rangle\langle 2| + \mathbf{D}^* |2\rangle\langle 1| \quad (19)$$

and  $\mathbf{E}(\mathbf{r} = 0)$  is the observable for the electric field at the position  $\mathbf{r} = 0$  of the atom given by

$$\mathbf{E}(\mathbf{r} = 0) = i \sum_{\mathbf{k}} \sum_{\lambda=1,2} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 L^3}} \left[ \epsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} - \epsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}^* \right]. \quad (20)$$

Here we assumed that the atom is so small that the position of its electron is always the same. As introduced in the previous lectures, we describe the atom by the Hamiltonian

$$H_{\text{atom}} = \frac{1}{2} \hbar\omega_0 (|2\rangle\langle 2| - |1\rangle\langle 1|) \quad (21)$$

and the free radiation field by the Hamiltonian

$$H_{\text{env}} = \sum_{\mathbf{k}} \sum_{\lambda=1,2} \hbar\omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}. \quad (22)$$

The total energy of the system is, in the Schrödinger picture, described by the Hamiltonian

$$H = H_{\text{atom}} + H_{\text{env}} + H_{\text{int}}. \quad (23)$$

As this is in general the case for observables in the Schrödinger picture, this operator is time independent. Nevertheless, it is advantageous to go over into the interaction picture with respect to the interaction free Hamiltonian. For reasons that will become obvious later, we choose  $H_0 = H_{\text{atom}} + H_{\text{env}}$  with

$$\begin{aligned} U_0(t, 0) &= \left[ |1\rangle\langle 1| \exp\left(\frac{i}{2}\omega_0 t\right) + |2\rangle\langle 2| \exp\left(-\frac{i}{2}\omega_0 t\right) \right] \cdot \exp\left(-i \sum_{\mathbf{k}} \sum_{\lambda=1,2} \omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}\right) \\ &= \left[ |1\rangle\langle 1| \exp\left(\frac{i}{2}\omega_0 t\right) + |2\rangle\langle 2| \exp\left(-\frac{i}{2}\omega_0 t\right) \right] \cdot \prod_{\mathbf{k},\lambda=1,2} \exp\left(-i\omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}\right). \end{aligned} \quad (24)$$

Using the relations

$$\begin{aligned} \exp(i\omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}) a_{\mathbf{k},\lambda} \exp(-i\omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}) &= a_{\mathbf{k},\lambda} e^{-i\omega_{\mathbf{k}} t}, \\ \exp(i\omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}) a_{\mathbf{k},\lambda}^\dagger \exp(-i\omega_{\mathbf{k}} a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda}) &= a_{\mathbf{k},\lambda}^\dagger e^{i\omega_{\mathbf{k}} t}, \end{aligned} \quad (25)$$

which can be proven by showing that the derivatives of both sides are the same and they also coincide for  $t = 0$ , we obtain the interaction Hamiltonian

$$\begin{aligned} H_I(t) &= U_0^\dagger(t, 0) H_{\text{int}} U_0(t, 0) \\ &= ie \sum_{\mathbf{k}} \sum_{\lambda=1,2} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 L^3}} U_0^\dagger(t, 0) \left[ \epsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} - \epsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}^* \right] \left( \mathbf{D} |1\rangle\langle 2| + \mathbf{D}^* |2\rangle\langle 1| \right) U_0(t, 0) \\ &= ie \sum_{\mathbf{k}} \sum_{\lambda=1,2} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 L^3}} \left[ \epsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} e^{-i\omega_{\mathbf{k}} t} - \epsilon_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}^* e^{i\omega_{\mathbf{k}} t} \right] \cdot \left( \mathbf{D} |1\rangle\langle 2| e^{-i\omega_0 t} + \mathbf{D}^* |2\rangle\langle 1| e^{i\omega_0 t} \right). \end{aligned} \quad (26)$$

As before, in the derivation of the laser Hamiltonian, we now apply the *rotating wave approximation* and neglect the fast oscillating terms, whose effect on the time evolution is known to be much smaller than the effect of the slowly oscillating terms. The strongest interaction between the atoms and the radiation field occurs for photon modes with  $\omega_{\mathbf{k}} \approx \omega_0$ . We therefore take only terms proportional  $e^{\pm i(\omega_0 - \omega_{\mathbf{k}})t}$  but not terms proportional  $e^{\pm i(\omega_0 + \omega_{\mathbf{k}})t}$  into account and find

$$H_I(t) = \hbar \sum_{\mathbf{k}} \sum_{\lambda=1,2} \left[ g_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} |2\rangle\langle 1| e^{i(\omega_0 - \omega_{\mathbf{k}})t} + g_{\mathbf{k},\lambda}^* a_{\mathbf{k},\lambda}^\dagger |1\rangle\langle 2| e^{-i(\omega_0 - \omega_{\mathbf{k}})t} \right] \quad (27)$$

with the atom-photon coupling constants

$$g_{\mathbf{k},\lambda} = ie \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 L^3}} \epsilon_{\mathbf{k},\lambda} \cdot \mathbf{D}^*. \quad (28)$$

In the next lecture, we use the Hamiltonian (27) to describe the spontaneous emission processes and to derive the quantum jump approach and the master equation formalism for the single atom considered here.