Quantum optics lecture
Part V: The quantum jump approach and the master equations

Almut Beige
(Dated: April 8, 2005)

Summary. This is the final lecture of this Quantum Optics course. In it, we will use the atom-field Hamiltonian obtained at the end of the previous lecture, to introduce the quantum jump approach and the master equations for the quantum optical description of a single laser-driven two-level atom. Although, this is a very special example, it contains the basic quantum optical tools for the analysis of a variety of systems, like atom-cavity systems, ion traps and others.

I. SPONTANEOUS PHOTON EMISSIONS

Aim of this lecture is to find an as accurate as possible description of the trapped atom placed inside the free radiation field and driven by a laser (see Figure 1). Experimental observations of this system show that the laser causes oscillations between level 1 and 2. We have analysed these Rabi oscillations already in the previous lecture in some detail. Moreover, at random times, the atom might jump back into its ground state while emitting a photon. The emitted photons can be registered by causing a click at a single photon detector or the walls of the laboratory. We already know that the statistics of the emitted photons shows antibunching. A possible trajectory of the atom, as it may be recorded by a single photon detector, is shown in Figure 1.

Up to now, we used the Hamiltonian $H_{\rm laser}$ to describe the effect of the laser on the time evolution of the populations of the atomic levels 1 and 2, namely P_1 and P_2, in the absence of spontaneous emission. To model the possible trajectories of such an atom, when it is able to emit photons at a high rate, we used Einstein's rate equation and assumed that the atom is always either in the ground or the excited state. In this lecture, we will see how to take both effects, namely the sudden jumps during photon emissions and the coherent evolution inbetween, within the same model into account. We derive the quantum jump approach and the master equations from basic quantum optical principles.

Let us consider again the trapped two-level atom. For simplicity, we assume that it is driven by a resonant laser field. Then the Hamiltonian is of the form

$$H = H_{\rm atom} + H_{\rm env} + H_{\rm int} + H_{\rm laser} \, , \tag{1}$$

where we used the same notation as in previous lectures. Going over into the interaction picture with respect to the interaction-free Hamiltonian $H_0 = H_{\rm atom} + H_{\rm env}$ we obtain the interaction Hamiltonian

$$\begin{aligned} H_{\rm I}(t) &= U_0^\dagger(t,0) \left(H_{\rm int} + H_{\rm laser} \right) U_0(t,0) \\ &= \tfrac{1}{2} \hbar \Omega \left(|2\rangle\langle 1| + |1\rangle\langle 2| \right) + \hbar \sum_{\mathbf{k}} \sum_{\lambda=1,2} \left[g_{\mathbf{k},\lambda} \, a_{\mathbf{k},\lambda} \, |2\rangle\langle 1| \, {\rm e}^{{\rm i}(\omega_0-\omega_k)t} + g^*_{\mathbf{k},\lambda} \, a^\dagger_{\mathbf{k},\lambda} \, |1\rangle\langle 2| \, {\rm e}^{-{\rm i}(\omega_0-\omega_k)t} \right] , \end{aligned} \tag{2}$$

where Ω is the Rabi frequency of the laser and $g_{\mathbf{k},\lambda}$ is the atom-photon coupling constant

$$g_{\mathbf{k},\lambda} = {\rm i} e \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 L^3}} \, \epsilon_{\mathbf{k},\lambda} \cdot \mathbf{D}^* \, . \tag{3}$$

The time evolution of this Hamiltonian alone cannot predict anything like the spontaneous emission of a photon. It can only describe the evolution of an initial state with some population in the excited atomic state and no photons in

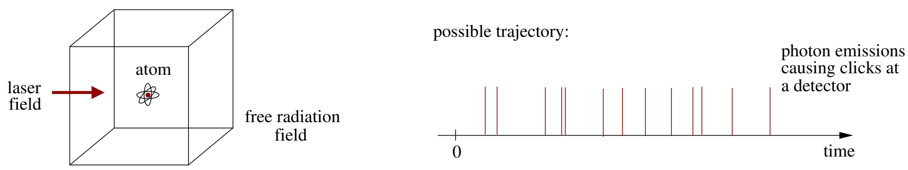

FIG. 1: A single laser-driven atom trapped inside a free radiation field and a possible trajectory, as it might be observed in a quantum optics experiment.

the free radiation field into a state with excitation accumulating in all possible photon modes $(\mathbf{k}, \lambda)$ and the ground state $|1\rangle$ and less population in the atomic state $|2\rangle$.

The origin for the spontaneous creation of a photon is the environment formed by the photon detector or the walls of the laboratory. In the following, we model the environment outside the box containing the free radiation field by assuming that it performs measurements at small time intervals Δt apart, whether a photon has been created or not [1]. If so, the photon is absorbed in the detection process and the radiation field is again in its vacuum state. This model is in good agreement with and mainly motivated by experimental findings in recent quantum optics experiments. It is based on the assumption that an emitted photon cannot re-excite the atom again. Once lost, it becomes part of the environment.

Suppose $|0_{\rm vac}\rangle$ denotes the vacuum state of the free radiation field and the initial state of the system equals at $t = 0$

$$|0_{\rm vac}\rangle|\varphi\rangle = |0_{\rm vac}\rangle \otimes \big(\alpha\, |1\rangle + \beta\, |2\rangle \big) \, . \tag{4}$$

Then the probability to find a photon in any of the possible modes $(\mathbf{k}, \lambda)$ after a short time Δt can be calculated using the usual quantum mechanical projector formalism for the description of the measurement process. Doing so we find

$$P_{\rm 1\,photon}(\Delta t) = \Big\| \sum_{\mathbf{k}} \sum_{\lambda=1,2} |1_{\mathbf{k},\lambda}\rangle\langle 1_{\mathbf{k},\lambda}|\, U_{\rm I}(\Delta t, 0)\, |0_{\rm vac}\rangle|\varphi\rangle \Big\|^2 \, , \tag{5}$$

where $|1_{\mathbf{k},\lambda}\rangle$ is the state with one photon in mode $(\mathbf{k}, \lambda)$. To calculate this probability up to first order in Δt, we use first order perturbation theory and obtain

$$\begin{aligned} P_{\rm 1\,photon}(\Delta t) &= \Big\| \sum_{\mathbf{k}} \sum_{\lambda=1,2} |1_{\mathbf{k},\lambda}\rangle\langle 1_{\mathbf{k},\lambda}| \Big(I\!I - \frac{\rm i}{\hbar} \int_0^{\Delta t} {\rm d}t\, H_{\rm I}(t) \Big)\, |0_{\rm vac}\rangle|\varphi\rangle \Big\|^2 \\ &= \Big\| \sum_{\mathbf{k}} \sum_{\lambda=1,2} |1_{\mathbf{k},\lambda}\rangle\, (-{\rm i}) \int_0^{\Delta t} {\rm d}t\, g^*_{\mathbf{k},\lambda}\, {\rm e}^{-{\rm i}(\omega_0-\omega_k)t}\, |1\rangle\langle 2|\varphi\rangle \Big\|^2 \\ &= |\beta|^2 \int_0^{\Delta t} {\rm d}t \int_0^{\Delta t} {\rm d}t' \sum_{\mathbf{k}} \sum_{\lambda=1,2} |g_{\mathbf{k},\lambda}|^2\, {\rm e}^{-{\rm i}(\omega_0-\omega_k)(t-t')} \, . \end{aligned} \tag{6}$$

To calculate this integral to a very good approximation, we substitute $\tau = t - t'$ and use the relation

$$\int_0^{\Delta t} {\rm d}\tau\, {\rm e}^{-{\rm i}(\omega_0-\omega_k)\tau} \approx \pi\, \delta(\omega_0 - \omega_k) \, , \tag{7}$$

such that Eq. (6) becomes

$$\begin{aligned} P_{\rm 1\,photon}(\Delta t) &= |\beta|^2 \int_0^{\Delta t} {\rm d}t \sum_{\mathbf{k}} \sum_{\lambda=1,2} |g_{\mathbf{k},\lambda}|^2\, \pi\, \delta(\omega_0 - \omega_k) \\ &\equiv |\beta|^2 \cdot \Gamma \cdot \Delta t \, . \end{aligned} \tag{8}$$

Here Γ is the spontaneous decay rate of the exited state $|2\rangle$ of the atom. A closer evaluation of the integral on the right hand side of the above equation would show that Γ equals

$$\Gamma = \frac{e^2 |\mathbf{D}|^2 \omega_0^3}{6\pi\epsilon_0 \hbar c^3} \, . \tag{9}$$

At this point it is important to note that the probability to find one photon in Δt is indeed proportional Δt as suggested by experiments and previously predicted by Einstein's rate equation model. The coupling of the two-level atom to a reservoir of inifinitely many harmonic oscillators, namely the different photon modes $(\mathbf{k}, \lambda)$, can indeed result in the sudden creation of a photon. The probability density for this to happen equals $\Gamma \cdot |\beta|^2 = \Gamma \cdot |\,\langle 2|\varphi\rangle\,|^2$. Of course, the photon emission is not exactly a spontaneous process. It requires a finite time Δt but since Δt is very short compared to all other typical times characterising the system, like $1/\Omega$ and $1/\Gamma$, the creation of a photon can be considered a quantum jump to a very good approximation. Finally, we remark that an atom is always prepared in state $|1\rangle$ immediately after an emission, as one can see from the form of the Hamiltonian (2) and the above calculations.

II. THE NO-PHOTON TIME EVOLUTION

We now already know the probability density for the emission of a photon given the initial atomic state $|\varphi\rangle$ and that such an event transfers the atom into its ground state. However, since Δt is relatively small, the measurements performed on the free radiation field result in most cases in the detection of *no* photon, i.e. the vacuum state $|0_{\rm vac}\rangle$. This detection of no photon also affects the time evolution of the atom, since it reveals some information about the atom. The longer no photon is emitted, the more likely it is that the atom is in its ground state. As a consequence, the no-photon time evolution damps away population in the excited state $|2\rangle$. Between jumps, the atom does *not* simply follow the usual Schrödinger equation.

Let us consider again the case, where the system is initially prepared in the state (4). Then the state of the system equals at a time Δt later under the condition of no photon emission

$$|0_{\rm vac}\rangle|\varphi^0(\Delta t)\rangle = |0_{\rm vac}\rangle\langle 0_{\rm vac}|\, U_{\rm I}(\Delta t,0)\,|0_{\rm vac}\rangle|\varphi\rangle = |0_{\rm vac}\rangle\, U_{\rm cond}(\Delta t,0)\,|\varphi\rangle \tag{10}$$

with the conditional non-unitary time evolution operator

$$U_{\rm cond}(\Delta t,0) \equiv \langle 0_{\rm vac}|\, U_{\rm I}(\Delta t,0)\,|0_{\rm vac}\rangle\,. \tag{11}$$

Introducing this time evolution operator will give us a very convinient tool for the description of the no-photon time evolution of the atom. As we see below, the atom evolves according to the Schrödinger equation given by the non-Hermitian Hamiltonian $H_{\rm cond}$, which we derive in the following and which is widely independent of the concrete choice of Δt. As in the previous Section, the definition of a concrete Δt is an auxiliary tool with no physical consequences.

Calculating the conditional time evolution operator using second order perturbation theory, we find

$$\begin{aligned} U_{\rm cond}(\Delta t,0) &= \langle 0_{\rm vac}|\left(I\!I - \frac{\rm i}{\hbar}\int_0^{\Delta t} {\rm d}t\, H_{\rm I}(t) - \frac{1}{\hbar^2}\int_0^{\Delta t}{\rm d}t\int_0^t {\rm d}t'\, H_{\rm I}(t)\,H_{\rm I}(t')\right)|0_{\rm vac}\rangle \\ &= I\!I - \frac{\rm i}{\hbar}\,H_{\rm laser}\,\Delta t - \frac{1}{\hbar^2}\int_0^{\Delta t}{\rm d}t\int_0^t {\rm d}t'\,\langle 0_{\rm vac}|\,H_{\rm I}(t)\,H_{\rm I}(t')\,|0_{\rm vac}\rangle\,. \end{aligned} \tag{12}$$

Using Eq. (2) and proceeding exactly as in the previous Section, the last term in this equation can be further simplified. With the help of Eqs. (7) and (8), we find

$$\begin{aligned} -\frac{1}{\hbar^2}\int_0^{\Delta t}{\rm d}t\int_0^t {\rm d}t'\,\langle 0_{\rm vac}|\,H_{\rm I}(t)\,H_{\rm I}(t')\,|0_{\rm vac}\rangle &= -\int_0^{\Delta t}{\rm d}t\int_0^t {\rm d}t'\sum_{\mathbf{k}}\sum_{\lambda=1,2}|g_{\mathbf{k},\lambda}|^2\,{\rm e}^{-{\rm i}(\omega_0-\omega_k)(t-t')}\,|2\rangle\langle 2| \\ &\approx -\tfrac{1}{2}\int_0^{\Delta t}{\rm d}t\int_0^{\Delta t}{\rm d}t'\sum_{\mathbf{k}}\sum_{\lambda=1,2}|g_{\mathbf{k},\lambda}|^2\,{\rm e}^{-{\rm i}(\omega_0-\omega_k)(t-t')}\,|2\rangle\langle 2| \\ &= -\tfrac{1}{2}\pi\sum_{\mathbf{k}}\sum_{\lambda=1,2}|g_{\mathbf{k},\lambda}|^2\,\delta(\omega_0-\omega_k)\cdot\Delta t\,|2\rangle\langle 2| \\ &= -\tfrac{1}{2}\Gamma\cdot\Delta t\,|2\rangle\langle 2|\,. \end{aligned} \tag{13}$$

In the following, $H_{\rm cond}$ denotes the conditional Hamiltonian defined by the equation

$$U_{\rm cond}(\Delta t,0) = I\!I - \frac{\rm i}{\hbar}\,H_{\rm cond}\,\Delta t + \mathcal{O}(\Delta t^2)\,. \tag{14}$$

A comparison of Eq. (14) with Eqs. (12) and (13) then yields

$$H_{\rm cond} = H_{\rm laser} - \frac{\rm i}{2}\hbar\,\Gamma\,|2\rangle\langle 2|\,. \tag{15}$$

The state of the atom under the condition of no photon emission can be calculated by solving the corresponding Schrödinger equation

$${\rm i}\hbar\,\frac{\rm d}{{\rm d}t}\,|\varphi^0(t)\rangle = H_{\rm cond}\,|\varphi^0(t)\rangle\,. \tag{16}$$

From Eq. (10) we see that the state $|\varphi^0(t)\rangle$ is in general not normalised. While evolving with $H_{\rm cond}$ the norm of the state vector $|\varphi^0(t)\rangle$ decreases in general in time such that

$$P_0(t) = \big\|\, U_{\rm cond}(t,0)\,|\varphi\rangle\,\big\|^2 \tag{17}$$

equals by construction (see Eq. (10)) the probability for no photon in $(t,0)$ given the initial state $|\varphi\rangle$.

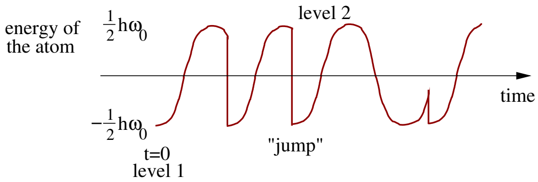

FIG. 2: Possible evolution of the energy of the trapped atom in the presence of spontaneous photon emission and resonant laser driving.

III. THE QUANTUM JUMP APPROACH

The quantum jump approach is the name for the method introduced in the previous two Sections. It was originally introduced simultaneously by Hegerfeldt and Wilder in Germany, Dalibard, Castin and Mølmer in France and Carmichael from New Zealand and can be used to predict the possible trajectories for the time evolution of the single laser driven two-level atom (see Figure 2). Although we should now be able to solve the Schrödinger equation (16) for this case explicitly, we will restrict ourselves here to the analysis of a simpler example.

We consider again the single two-level atom prepared in the initial state (4) but neglect the presence of any laser fields. Then the conditional Hamiltonian (15) simplifies to the operator

$$H_{\rm cond} = -\frac{\rm i}{2}\hbar\,\Gamma\,|2\rangle\langle 2| \tag{18}$$

with the corresponding time evolution operator

$$U_{\rm cond}(t,0) = |1\rangle\langle 1| + \exp\big(-\tfrac{1}{2}\Gamma t\big)\,|2\rangle\langle 2|\,. \tag{19}$$

Using the quantum jump approach, we can calculate the state $|\varphi^0(t)\rangle$ of the atom at any time t under the condition of no photon emission by applying $U_{\rm cond}(t,0)$ to the initial state (4),

$$|\varphi^0(t)\rangle = \alpha\,|1\rangle + \exp\big(-\tfrac{1}{2}\Gamma t\big)\,\beta\,|2\rangle\,. \tag{20}$$

The probability for no photon emission in $(0,t)$ equals

$$P_0(t) = |\,\alpha\,|^2 + \exp\big(-\Gamma t\big)\,|\,\beta\,|^2\,, \tag{21}$$

which becomes $|\,\alpha\,|^2$ for long enough times t and $t\to\infty$. As expected, the initial population in the excited state $|2\rangle$ decays exponentially and the mean time for the emission of the photon equals $1/\Gamma$. (You can check this.) One can also show that the probability density for the emission of a photon is at any time t given by the actual population in the excited state multiplied with Γ, i.e. $|\,\langle 2|\varphi^0(t)\rangle\,|^2/\|\,|\varphi^0(t)\rangle\,\|^2\cdot\Gamma$. In case of an emission, the atom jumps into the ground state $|1\rangle$ and the no-photon evolution becomes trivial.

IV. THE MASTER EQUATIONS

To describe an ensemble of many one-atom systems or to calculate the stationary state of a single atom, it is convenient to use the so-called master equations. They are a standard methods used in quantum optics for the description of open quantum systems. In the following, we describe such an ensemble by the density matrix operator ρ as it has been introduced in the first lecture (see Section II B). The reason for this is that each atom of the ensemble emits photons at different random times so that, after a short time, all atoms are prepared in a different pure state. In case of a single atom, using a density matrix description expresses the fact that it is not known, when the last photon has been emitted.

Suppose the state of the ensemble equals at a time $t=0$ the density matrix of a pure state and

$$\rho(0) = |\varphi\rangle\langle\varphi|\,. \tag{22}$$

Then we have at a short time Δt later

$$\rho(\Delta t) = P_0(t)\,\rho(\text{subensemble with no photons}) + \Gamma\,\Delta t\,\rho(\text{subensemble with one photon})\,, \tag{23}$$

which equals

$$\rho(\Delta t) = U_{\rm cond}(\Delta t, 0)\,|\varphi\rangle\langle\varphi|\,U^\dagger_{\rm cond}(\Delta t, 0) + \Gamma\,\Delta t\,|\langle\varphi|2\rangle|^2\,|1\rangle\langle 1|$$
$$= U_{\rm cond}(\Delta t, 0)\,|\varphi\rangle\langle\varphi|\,U^\dagger_{\rm cond}(\Delta t, 0) + \Gamma\,\Delta t\,|1\rangle\langle 2|\varphi\rangle\langle\varphi|2\rangle\langle 1|\,. \tag{24}$$

In general, i.e. independent of the form of the initial state, one has

$$\rho(\Delta t) = U_{\rm cond}(\Delta t, 0)\,\rho(0)\,U^\dagger_{\rm cond}(\Delta t, 0) + \Gamma\,\Delta t\,|1\rangle\langle 2|\,\rho(0)\,|2\rangle\langle 1|\,. \tag{25}$$

Taking the derivation of ρ,

$$\frac{{\rm d}}{{\rm d}t}\rho(\Delta t) = \lim_{\Delta t \to 0} \frac{\rho(\Delta t) - \rho(0)}{\Delta t}\,, \tag{26}$$

we find that the time evolution of the density matrix is governed by the *master equation*

$$\frac{{\rm d}}{{\rm d}t}\rho = -\frac{{\rm i}}{\hbar}\left[H_{\rm cond}\,\rho - \rho\,H^\dagger_{\rm cond}\right] + \Gamma\,|1\rangle\langle 2|\,\rho\,|2\rangle\langle 1|\,. \tag{27}$$

This is a differential equation for the density matrix operator ρ of the so-called Lindblatt form. Lindblad equations also used in other areas of physics. Starting from a purely mathematical point of view, it can be shown that the time evolution of ρ has to be of a certain form in order to preserve its normalisation and positivity. Depending on the context, the damping constants like Γ may have a different interpretation.

Finally, we consider the stationary state of the atom, which can be calculated with the help of Eq. (27) since it obeys the condition

$$\frac{{\rm d}}{{\rm d}t}\rho = 0\,. \tag{28}$$

Using the above introduced conditional Hamiltonian (15) and the laser Hamiltonian $H_{\rm laser} = \frac{1}{2}\hbar\Omega\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)$ and solving a system of four linear equations, we find that the steady state population in ground and excited state equals

$$P_1 = \frac{\Gamma^2 + \Omega^2}{\Gamma^2 + 2\Omega^2} \quad \text{and} \quad P_2 = \frac{\Omega^2}{\Gamma^2 + 2\Omega^2}\,. \tag{29}$$

A comparison with Eq. (12) of the previous lecture notes (part IV) shows that this is in very good agreement with the predictions of Einstein's rate equations, if we identify Γ with the rate A and Ω with the rate B. These equations already model the behaviour of the system considered in the second half of this lecture very well. However, there are many examples, where a more detailed analysis of the time evolution of a system, using either the quantum jump approach or the master equations, is required to explain or predict the findings of quantum optical experiments.

Acknowledgement. I thank G. C. Hegerfeldt, who taught me a lot about Quantum Optics, and H. E. Brandt, who kindly pointed out mistakes in previous versions of these lecture notes.

---

[1] We cannot assume that the environment performs continuous measurements on the free radiation field, since such measurements on a quantum mechanical system would have the effect that the system remains trapped in its initial state. This is called the quantum Zeno effect (see Part I of the lecture notes) and not in agreement with the observation of photons in the real world. In the quantum Zeno or continuous measurement regime, the atom would not be able to generate a photon at all.